

LOCAL ALGEBRAIC EFFECT THEORIES

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Local Algebraic Effect Theories

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Abstract

Algebraic effects are computational effects that can be described with a set of basic operations and equations between them. As many interesting effect handlers do not respect these equations, most approaches assume a trivial theory, sacrificing both reasoning power and safety.

We present an alternative approach where the type system tracks equations that are observed in subparts of the program, yielding a sound and flexible logic, and paving a way for practical optimizations and reasoning tools.

Algebraic effects are computational effects that can be described by a *signature* of primitive operations and a collection of equations between them (Plotkin & Power, 2001; Plotkin & Power, 2003), while algebraic effect *handlers* are a generalization of exception handlers to arbitrary algebraic effects (Plotkin & Pretnar, 2009; Plotkin & Pretnar, 2013). Even though the early work considered only handlers that respect equations of the effect theory, a considerable amount of useful handlers did not, and the restriction was dropped in most — though not all (Ahman, 2018) — of the later work on handlers (Kammar *et al.*, 2013; Bauer & Pretnar, 2015; Leijen, 2017; Biernacki *et al.*, 2018), resulting in a weaker reasoning logic and imprecise specifications.

Our aim is to rectify this by reintroducing effect theories into the type system, tracking equations observed in parts of a program. On one hand, the induced logic allows us to rewrite computations into equivalent ones with respect to the effect theory, while on the other hand, the type system enforces that handlers preserve equivalences, further specifying their behaviour. After an informal overview in Section 1, we proceed as follows:

- The syntax of the working language, its operational semantics, and the typing rules are given in Section 2.
- Determining if a handler respects an effect theory is in general undecidable (Plotkin & Pretnar, 2013), so there is no canonical way of defining such a judgement. Therefore, the typing rules are given parametric to a reasoning logic, and in Section 3, we present some of the more interesting choices.
- Since the definition of typing judgements is intertwined with a reasoning logic, we must be careful when defining the denotation of types and terms. Thus, in Section 4, we first introduce a set-based denotational semantics that disregards effect theories and prove the expected meta-theoretic properties.

* This material is based upon work supported by the Air Force Office of Scientific Research under award number FA9550-17-1-0326.

PART I

ALGEBRAIC EFFECTS

every **computation**

either

returns a **value**

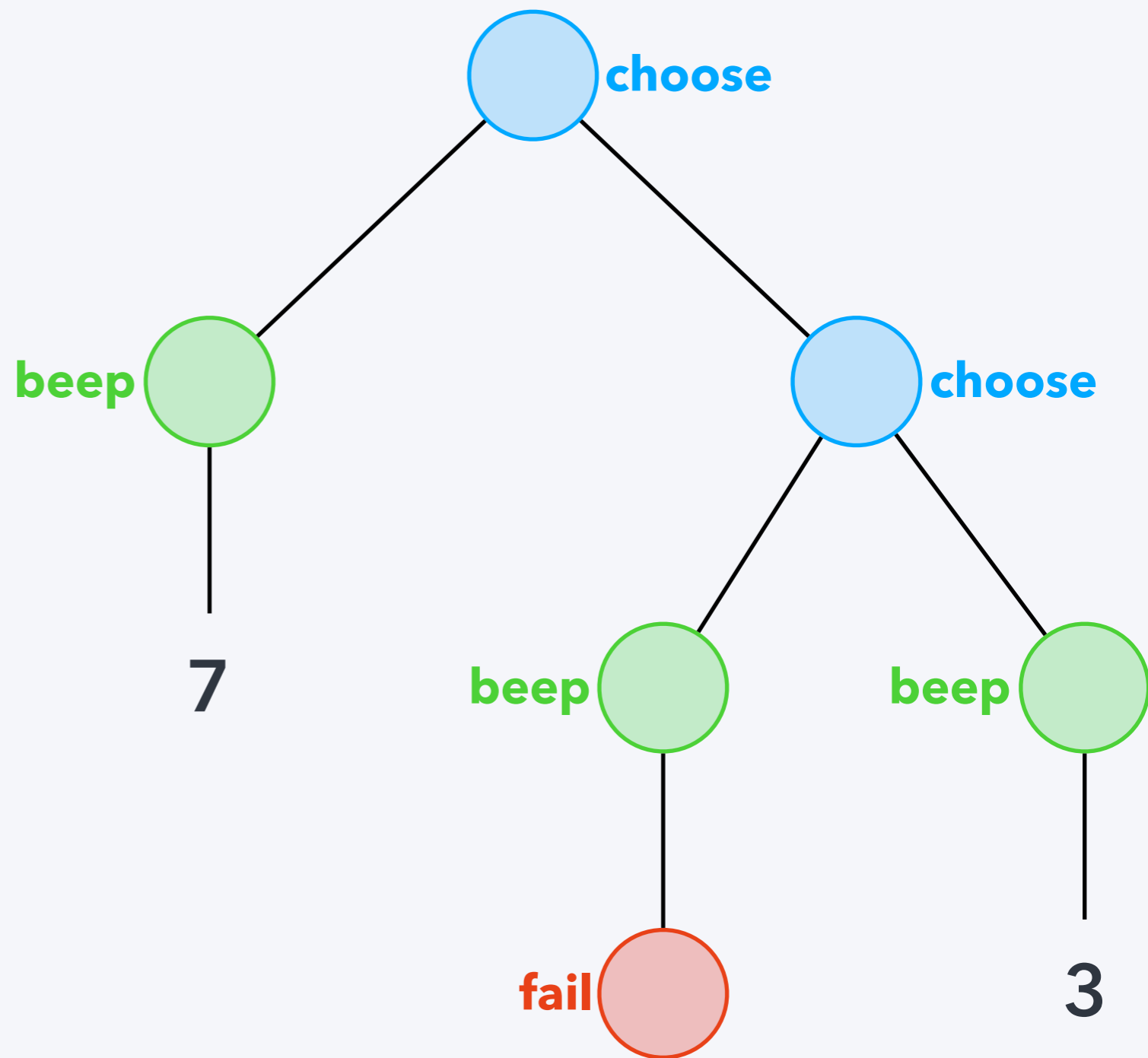
or

calls an **operation**

```

let divide m n =
  beep (); m / n
in
let x = choose 42 12 in
if x > 20 then
  divide x 6
else
  divide x (choose 0 4)

```



$\oplus : 2$ $\text{beep} : 1$ $\text{fail} : 0$

$$z \oplus z = z$$

$$z_1 \oplus z_2 = z_2 \oplus z_1$$

$$(z_1 \oplus z_2) \oplus z_3 = z_1 \oplus (z_2 \oplus z_3)$$

$$\text{beep}(z_1) \oplus \text{beep}(z_2) = \text{beep}(z_1 \oplus z_2)$$

$$\text{fail}() \oplus \text{fail}() = \text{fail}()$$

$\oplus : 2$ $\text{beep} : 1$ $\text{fail} : 0$

$$z \oplus z = z$$

$$z_1 \oplus z_2 = z_2 \oplus z_1$$

$$(z_1 \oplus z_2) \oplus z_3 = z_1 \oplus (z_2 \oplus z_3)$$

$$\text{beep}(z_1) \oplus \text{beep}(z_2) = \text{beep}(z_1 \oplus z_2)$$

$$\text{fail}() \oplus \text{fail}() = \text{fail}()$$

signature

$$\Sigma ::= \{\text{op}_1 : k_1, \dots, \text{op}_n : k_n\}$$

$$T ::= z \mid \text{op}(T_1, \dots, T_n)$$

$$\mathcal{E} ::= \{T_1 = T'_1, \dots, T_n = T'_n\}$$

effect theory

signature

$$\Sigma ::= \{\text{op}_i : k_i\}_i$$

$$T ::= z \mid \text{op}(T_i)_i$$

$$\mathcal{E} ::= \{T_i = T'_i\}_i$$

effect theory

values

$$v ::= x \mid () \mid \mathbf{fun} \ x \mapsto c$$
$$c ::= \mathbf{ret} \ v \mid \mathbf{do} \ x \leftarrow c_1 \ \mathbf{in} \ c_2 \mid v_1 \ v_2 \mid$$
$$\text{op}(c_i)_i$$

computations

$$A ::= \mathbf{unit} \mid A \rightarrow \underline{C}$$
$$\underline{C} ::= A! \{ \text{op}_i : k_i \}$$

$$v ::= x \mid () \mid \mathbf{fun} \ x \mapsto c$$
$$c ::= \mathbf{ret} \ v \mid \mathbf{do} \ x \leftarrow c_1 \ \mathbf{in} \ c_2 \mid v_1 \ v_2 \mid$$
$$\text{op}(c_i)_i$$

value types

$$A ::= \mathbf{unit} \mid A \rightarrow \underline{C}$$
$$\underline{C} ::= A! \{ \text{op}_i : k_i \}$$

computation types

value typing

$$x_1 : A_1, \dots, x_n : A_n \vdash v : A$$

$$x_1 : A_1, \dots, x_n : A_n \vdash c : \underline{C}$$

computation typing

$$(x : A) \in \Gamma$$

$$\Gamma \vdash x : A$$

$$\Gamma \vdash () : \text{unit}$$
$$\Gamma, x : A \vdash c : \underline{C}$$

$$\Gamma \vdash \mathbf{fun} \ x \mapsto c : A \rightarrow \underline{C}$$

$$\frac{\Gamma \vdash v_1 : A \rightarrow \underline{C} \quad \Gamma \vdash v_2 : A}{\Gamma \vdash v_1 v_2 : \underline{C}}$$

$$\frac{\Gamma \vdash v : A}{\Gamma \vdash \mathbf{ret} \, v : A ! \Sigma}$$

$$\Gamma \vdash c_1 : A ! \Sigma \quad \Gamma, x : A \vdash c_2 : B ! \Sigma$$

$$\Gamma \vdash \mathbf{do} \, x \leftarrow c_1 \, \mathbf{in} \, c_2 : B ! \Sigma$$

$$\left[\Gamma \vdash c_i : A ! \Sigma \right]_i \quad \text{op}_i : k_i \in \Sigma$$

$$\Gamma \vdash \text{op}(c_i)_i : A ! \Sigma$$

$$\Gamma \vdash v =_A v'$$

$$\Gamma \vdash c =_{\underline{C}} c'$$

$$x =_{\text{unit}} ()$$

$$(\mathbf{fun} \ x \mapsto c) v =_{\underline{C}} c[v/x]$$

$$\mathbf{fun} \ x \mapsto v \ x =_{A \rightarrow \underline{C}} v$$

do $x \leftarrow$ **ret** v **in** $c =_{\underline{C}} c[v/x]$

do $x \leftarrow \text{op}(c_i)_i$ **in** c
 $=_{\underline{C}}$
 $\text{op}(\text{do } x \leftarrow c_i \text{ in } c)_i$

standard congruence equations

$$\frac{(T = T') \in \mathcal{E}_{\text{global}} \quad [c_i : \underline{C}]_i}{T[c_i/z_i]_i =_{\underline{C}} T'[c_i/z_i]_i}$$

every **computation**

either

returns a **value**

or

calls an **operation**

$$\forall v : A. \phi(\mathbf{ret} \ v)$$

$$\left[\forall c_i : A ! \Sigma. \bigwedge_i \phi(c_i) \Rightarrow \phi(\mathbf{op}(c_i)_i) \right]_{\mathbf{op}_i : k_i \in \Sigma}$$

$$\forall c : A ! \Sigma. \phi(c)$$

do $x \leftarrow c$ in ret $x = c$

do $x_1 \leftarrow c_1$ in (do $x_2 \leftarrow c_2$ in c)

=

do $x_2 \leftarrow$ (do $x_1 \leftarrow c_1$ in c_2) in c

for $\Sigma = \{\oplus : 2, \text{beep} : 1\}$, we have:

$$\begin{aligned} & \mathbf{do} \ x \leftarrow c \ \mathbf{in} \ (c_1 \oplus c_2) \\ & \quad = \\ & (\mathbf{do} \ x \leftarrow c \ \mathbf{in} \ c_1) \oplus (\mathbf{do} \ x \leftarrow c \ \mathbf{in} \ c_2) \end{aligned}$$

$$\begin{aligned} & \mathbf{do} \ x \leftarrow c_1 \ \mathbf{in} \ \text{beep}(c_2) \\ & \quad = \\ & \text{beep}(\mathbf{do} \ x \leftarrow c_1 \ \mathbf{in} \ c_2) \end{aligned}$$

$$\begin{aligned} & \mathbf{do} \ x_1 \leftarrow c_1 \ \mathbf{in} \ (\mathbf{do} \ x_2 \leftarrow c_2 \ \mathbf{in} \ c) \\ & \quad = \\ & \mathbf{do} \ x_2 \leftarrow c_2 \ \mathbf{in} \ (\mathbf{do} \ x_1 \leftarrow c_1 \ \mathbf{in} \ c) \end{aligned}$$

Algebraic Foundations for Effect-Dependent Optimisations

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Abstract

We present a general theory of Gifford-style type and effect annotations, where effect annotations are sets of effects. Generality is achieved by recourse to the theory of algebraic effects, a development of Moggi's monadic theory of computational effects that emphasises the operations causing the effects at hand and their equational theory. The key observation is that annotation effects can be identified with operation symbols.

We develop an annotated version of Levy's Call-by-Push-Value language with a kind of computations for every effect set; it can be thought of as a sequential, annotated intermediate language. We develop a range of validated optimisations (i.e., equivalences), generalising many existing ones and adding new ones. We classify these optimisations as structural, algebraic, or abstract: structural optimisations always hold; algebraic ones depend on the effect theory at hand; and abstract ones depend on the global nature of that theory (we give modularly-checkable sufficient conditions for their validity).

Categories and Subject Descriptors D.3.4 [Processors]: Compilers; Optimization; F.3.1 [Specifying and Verifying and Reasoning about Programs]: Logics of programs; F.3.2 [Semantics of Programming Languages]: Algebraic approaches to semantics; Denotational semantics; Program analysis; F.3.3 [Studies of Program Constructs]: Type structure

General Terms Languages, Theory.

Keywords Call-by-Push-Value, algebraic theory of effects, code transformations, compiler optimisations, computational effects, denotational semantics, domain theory, inequational logic, relevant and affine monads, sum and tensor, type and effect systems, universal algebra.

1. Introduction

In Gifford-style type and effect analysis [27], each term of a programming language is assigned a type and an effect set. The type describes the values the term may evaluate to; the effect set describes the effects the term may cause during its computation, such as memory assignment, exception raising, or I/O.

For example, consider the following term M :

if true then $x := 1$ else $x := \text{deref}(y)$

It has unit type 1 as its sole purpose is to cause side effects; it has effect set $\{\text{update}, \text{lookup}\}$, as it might cause memory updates or look-ups. Type and effect systems commonly convey this information via a type and effect judgement:

$x : \text{Loc}, y : \text{Loc} \vdash M : 1 ! \{\text{update}, \text{lookup}\}$

The information gathered by such effect analyses can be used to guarantee implementation correctness¹, to prove authenticity properties [15], to aid resource management [44], or to optimise code using transformations. We focus on the last of these. As an example, purely functional code can be executed out of order:

$x \leftarrow M_1; y \leftarrow M_2; N = y \leftarrow M_2; x \leftarrow M_1; N$

This reordering holds more generally, if the terms M_1 and M_2 have non-interfering effects. Such transformations are commonly used in optimising compilers. They are traditionally called *optimisations*, even if neither side is always the more optimal.

In a sequence of papers, Benton et al. [4–8] prove soundness of such optimisations for increasingly complex sets of effects. However, any change in the language requires a complete reformulation of its semantics and so of the soundness proofs, even though the essential reasons for the validity of the optimisations remain the same. Thus, this approach is not robust, as small language changes cause global theory changes.

A possible way to obtain robustness is to study effect systems in general. One would hope for a modular approach, seeking to isolate those parts of the theory that change under small language changes, and then recombining them with the unchanging parts. Such a theory may not only be important for compiler optimisations in big, stable languages. It can also be used for effect-dependent equational reasoning. This use may be especially helpful in the case of small, domain-specific languages, as optimising compilers are hardly ever designed for them and their diversity necessitates proceeding modularly.

The only available general work on effect systems seems to be that of Marino and Millstein [28]. They devise a methodology to derive type and effect frameworks which they apply to a call-by-value language with recursion and references; however, their methodology does not account for effect-dependent optimisations.

Fortunately, Wadler and Thiemann [46, 47] had previously made an important connection with the monadic approach to computational effects. They translated judgements of the form $\Gamma \vdash M : A ! \varepsilon$ in a region analysis calculus to judgements of the form $\Gamma' \vdash M' : T_\varepsilon A$ in a multi-monadic calculus. They gave the latter calculus an operational semantics, and conjectured the existence of a corresponding general monadic denotational semantics in which T_ε would denote a monad corresponding to the effects in ε , and in which the partial order of effect sets and inclusions would

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¹E. Cooper, S. Lindley, P. Wadler, and J. Yallop. Links 0.5, 2009. <http://groups.inf.ed.ac.uk/links>.

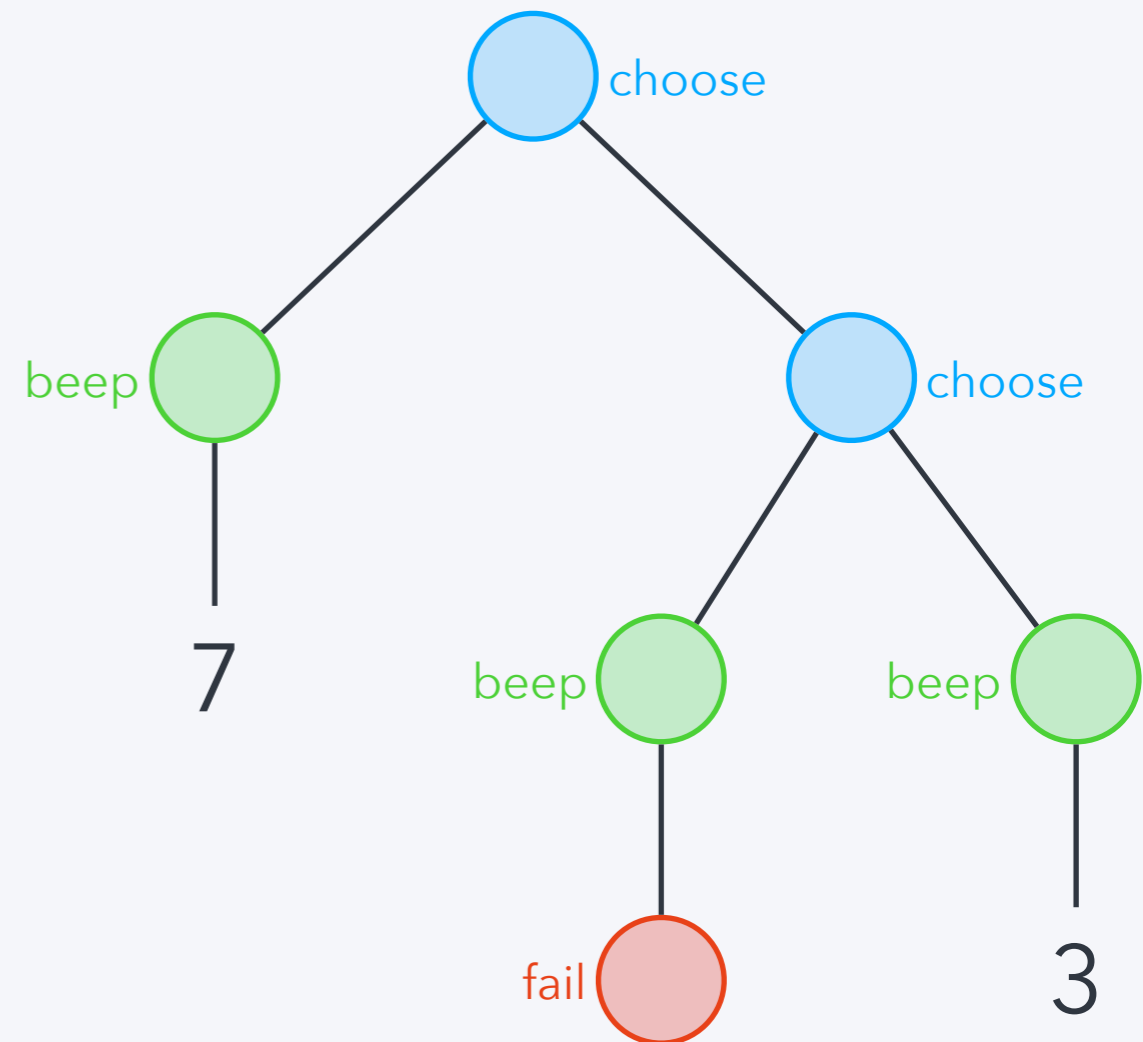
PART II

EFFECT **HANDLERS**

```

let divide m n =
  beep (); m / n
in
let x = choose 42 12 in
if x > 20 then
  divide x 6
else
  divide x (choose 0 4)

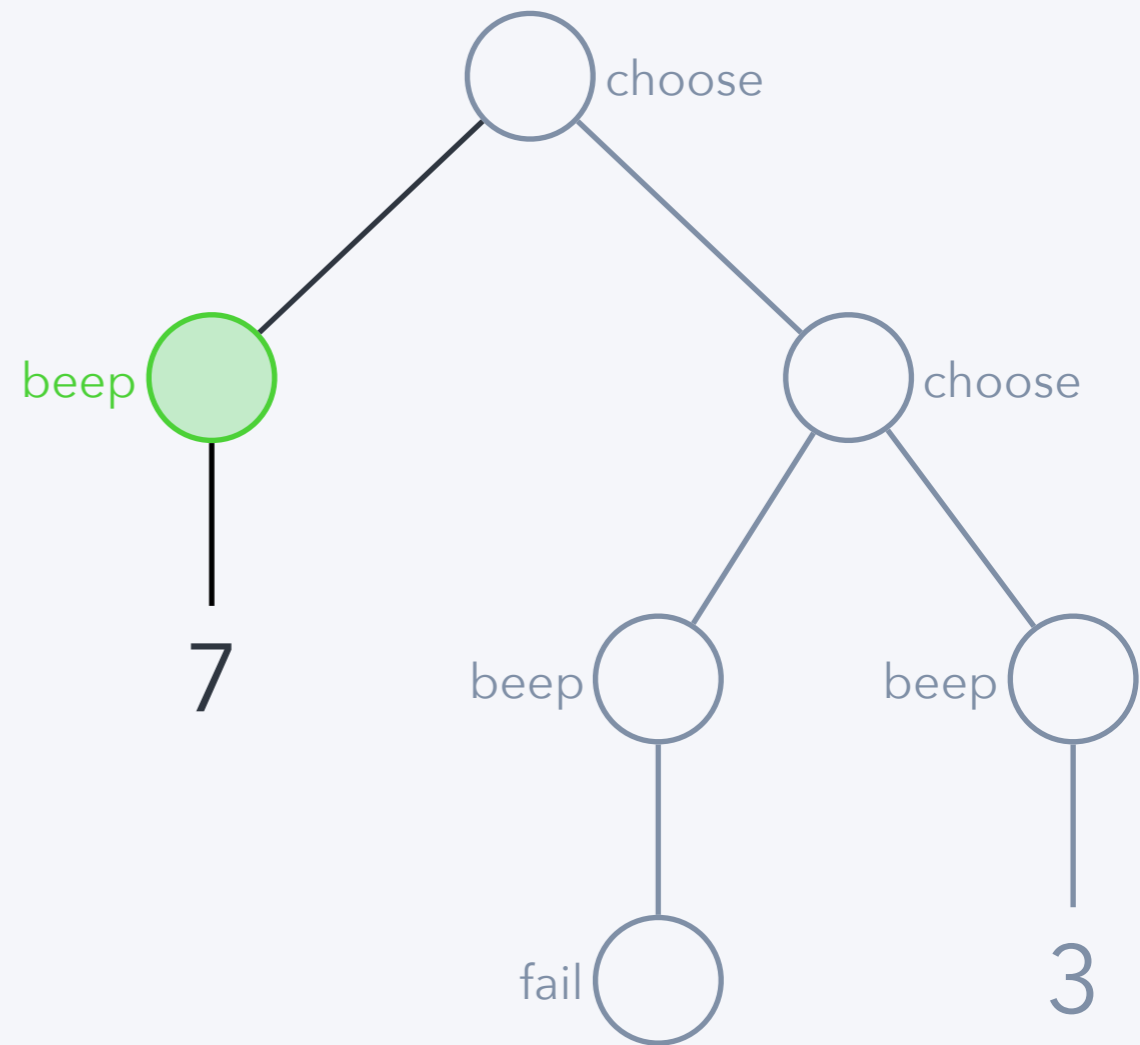
```



```

let goLeft = handler
  choose k1 k2 → k1 ()
in
with goLeft handle
  let divide m n =
    beep (); m / n
  in
  let x = choose 42 12 in
  if x > 20 then
    divide x 6
  else
    divide x (choose 0 4)

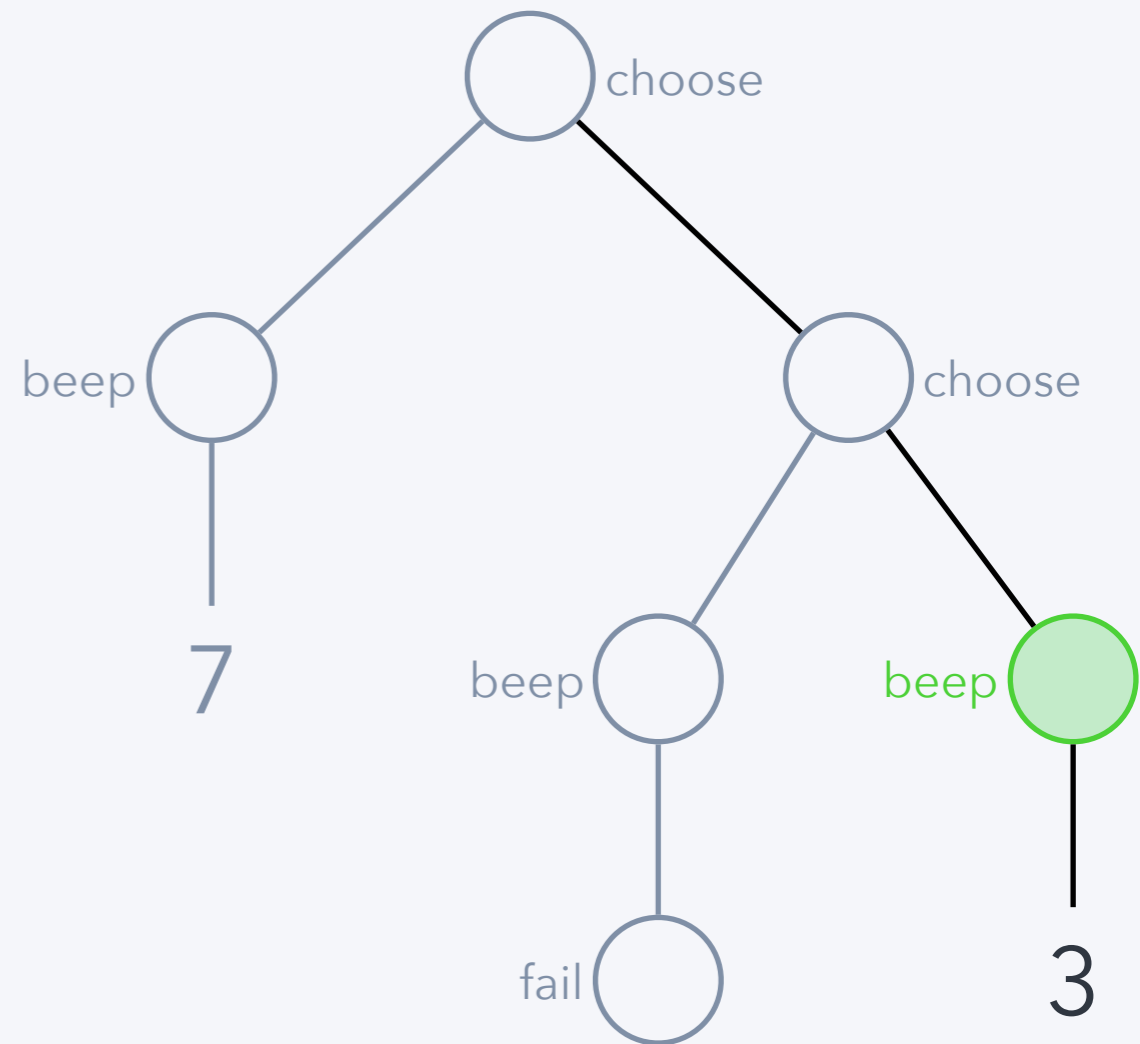
```



```

let goRight = handler
  choose  $k_1$   $k_2 \rightarrow k_2$  ()
in
with goRight handle
  let divide  $m$   $n$  =
    beep ();  $m / n$ 
  in
  let  $x$  = choose 42 12 in
  if  $x > 20$  then
    divide  $x$  6
  else
    divide  $x$  (choose 0 4)

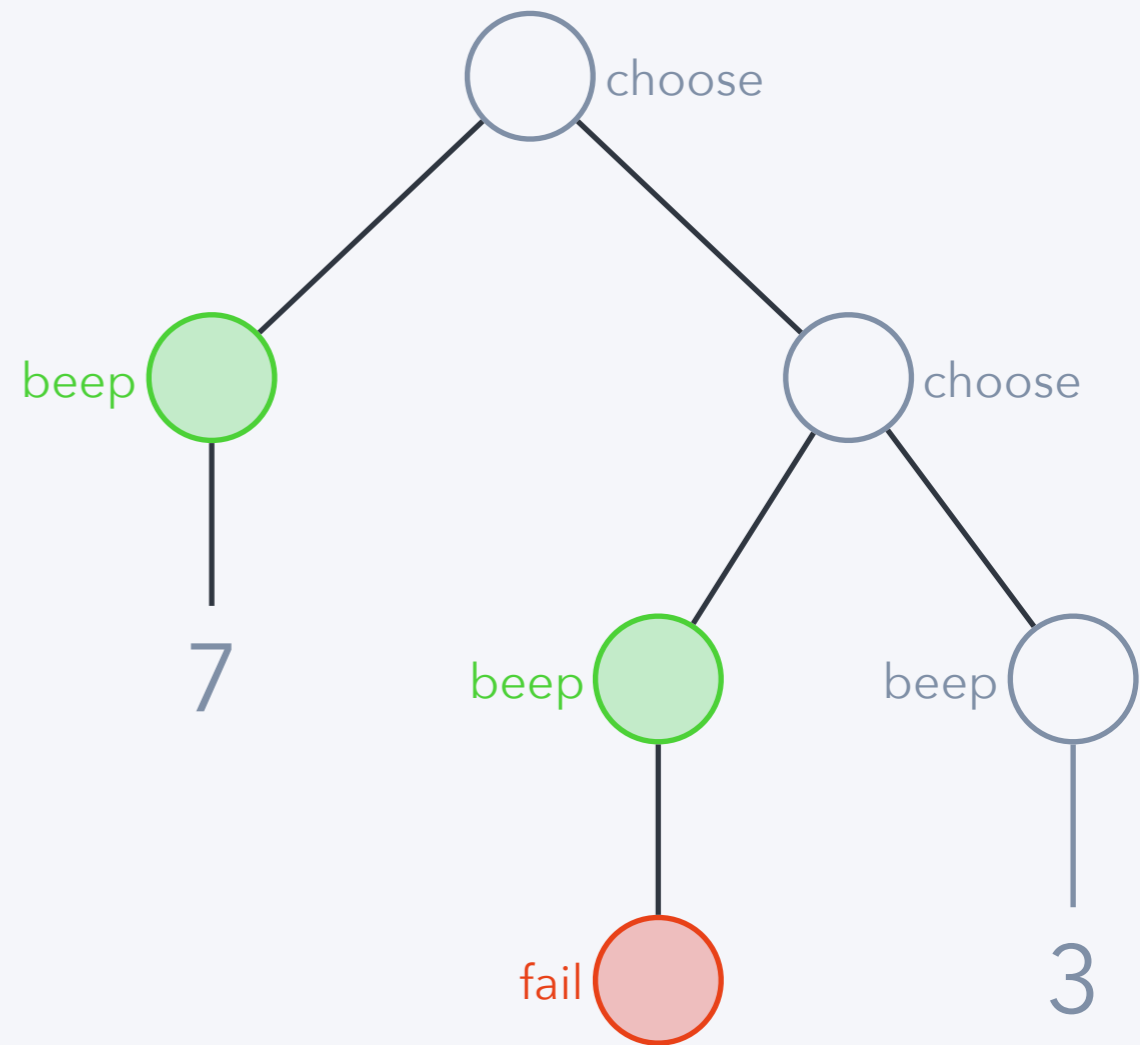
```



```

let pickMax = handler
  choose k1 k2 →
    max (k1 ()) (k2 ())
in
with pickMax handle
  let divide m n =
    beep (); m / n
  in
  let x = choose 42 12 in
  if x > 20 then
    divide x 6
  else
    divide x (choose 0 4)

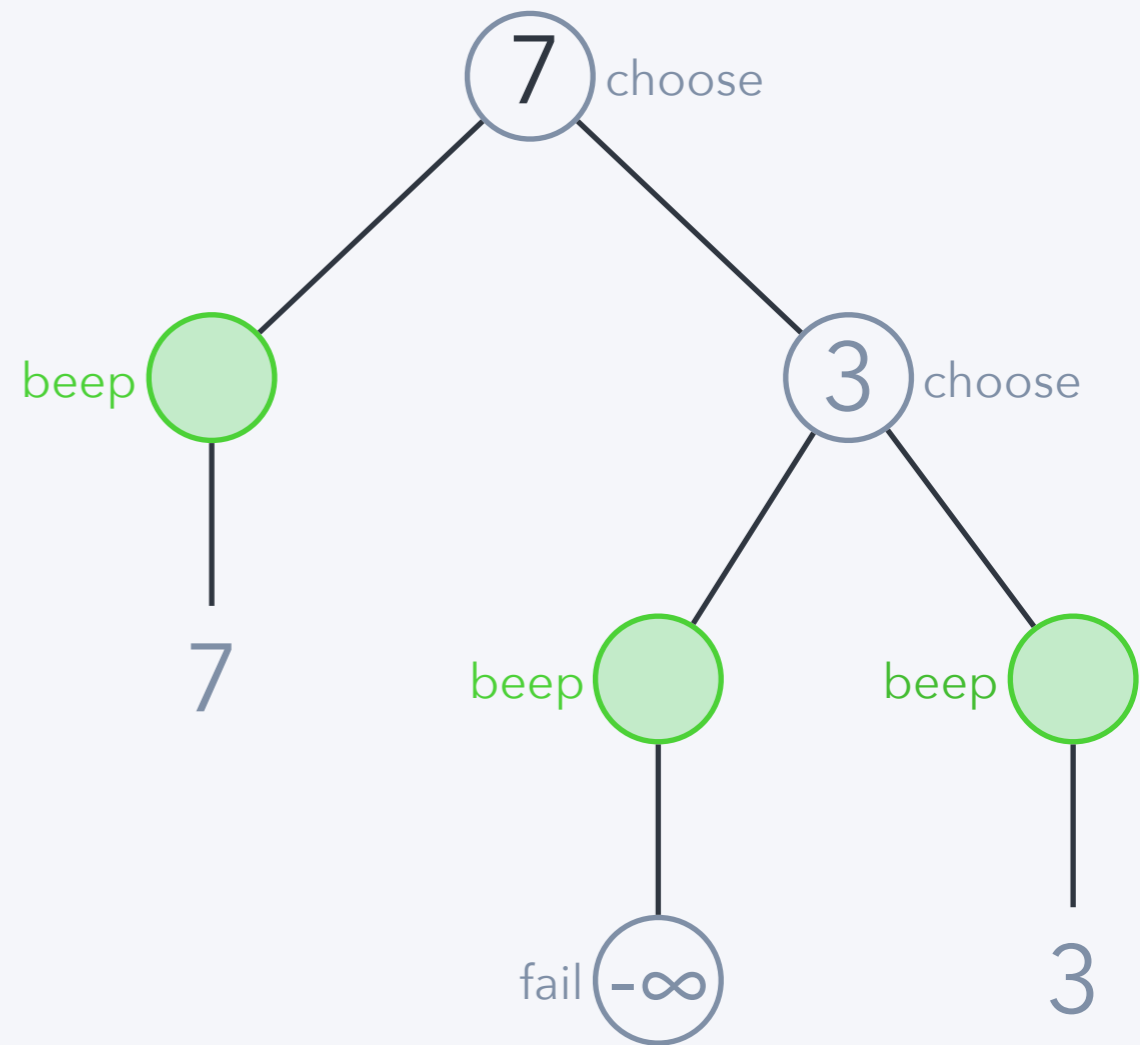
```



```

let pickMax = handler
  choose k1 k2 →
    max (k1 ()) (k2 ())
  fail → -inf
in
with pickMax handle
  let divide m n =
    beep (); m / n
  in
  let x = choose 42 12 in
  if x > 20 then
    divide x 6
  else
    divide x (choose 0 4)

```



let *sumAll* = **handler**

choose k_1 $k_2 \rightarrow$

$k_1 () + k_2 ()$

fail $\rightarrow 0$

beep $k \rightarrow k ()$

in

with *sumAll* **handle**

let *divide* m $n =$

$\text{beep } (); m / n$

in

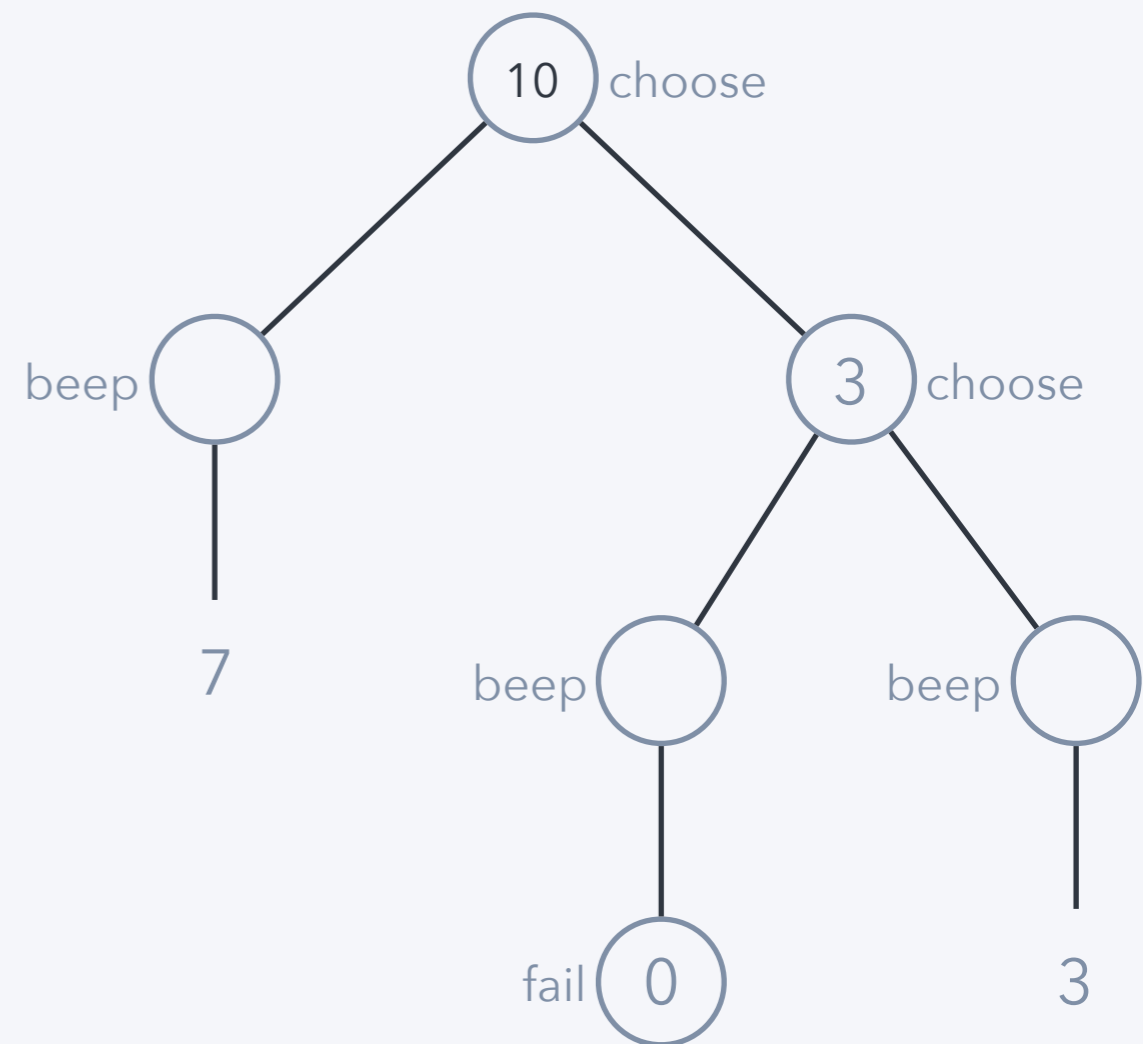
let $x = \text{choose } 42 \ 12$ **in**

if $x > 20$ **then**

$\text{divide } x \ 6$

else

$\text{divide } x \ (\text{choose } 0 \ 4)$



let *toList* = **handler**

choose k_1 $k_2 \rightarrow$

$k_1 () ++ k_2 ()$

fail $\rightarrow []$

beep $k \rightarrow k ()$

ret $x \rightarrow [x]$

in

with *toList* **handle**

let *divide* m $n =$

$\text{beep } (); m / n$

in

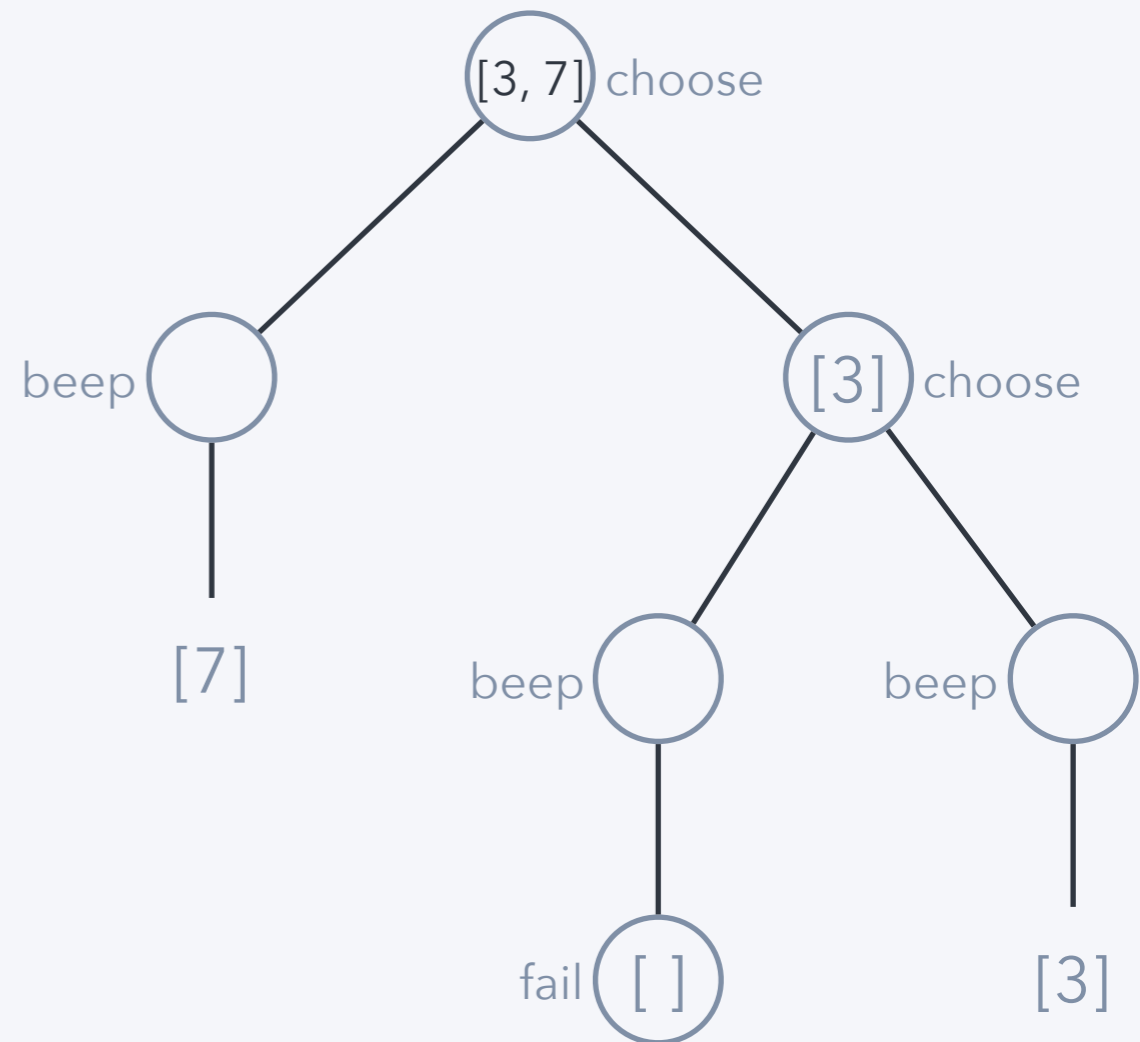
let $x = \text{choose } 42 \ 12$ **in**

if $x > 20$ **then**

$\text{divide } x \ 6$

else

$\text{divide } x \ (\text{choose } 0 \ 4)$



$v ::= \dots \mid \mathbf{handler} \, h$

$c ::= \dots \mid \mathbf{with} \, v \, \mathbf{handle} \, c$

$h ::= \{ \mathbf{ret} \, x \mapsto c, [\mathbf{op}_j(k_i)_i \mapsto c_j]_j \}$

$A ::= \dots \mid \underline{C}_1 \Rightarrow \underline{C}_2$

$\underline{C} ::= \dots$

$v ::= \dots \mid \mathbf{handler} \, h$

$c ::= \dots \mid \mathbf{with} \, v \, \mathbf{handle} \, c$

$h ::= \{ \mathbf{ret} \, x \mapsto c, [\mathbf{op}_j(k_i)_i \mapsto c_j]_j \}$

handlers

$A ::= \dots \mid \underline{C}_1 \Rightarrow \underline{C}_2$

$\underline{C} ::= \dots$

$v ::= \dots \mid \mathbf{handler} \, h$

$c ::= \dots \mid \mathbf{with} \, v \, \mathbf{handle} \, c$

$h ::= \{\mathbf{ret} \, x \mapsto c, [\mathbf{op}_j(k_i)_i \mapsto c_j]_j\}$

value types

$A ::= \dots \mid \underline{C}_1 \Rightarrow \underline{C}_2$

$\underline{C} ::= \dots$

computation types

value & computation typing

$$\Gamma \vdash v : A$$

$$\Gamma \vdash c : \underline{C}$$

$$\Gamma \vdash h : A ! \Sigma \Rightarrow \underline{C}$$

handler typing

$$\frac{\Gamma \vdash h : A ! \Sigma \Rightarrow \underline{C}}{\Gamma \vdash \mathbf{handler} \, h : A ! \Sigma \Rightarrow \underline{C}}$$

$$\frac{\Gamma \vdash v : \underline{C}_1 \Rightarrow \underline{C}_2 \quad \Gamma \vdash c : \underline{C}_1}{\Gamma \vdash \mathbf{with} \, v \, \mathbf{handle} \, c : \underline{C}_2}$$

$$\begin{array}{c}
\Gamma, x : A \vdash c : \underline{C} \\
[\Gamma, (k_i : \text{unit} \rightarrow \underline{C})_i \vdash c_j : \underline{C}]_j \\
\hline
\Gamma \vdash h : A ! \{\text{op}_j\}_j \Rightarrow \underline{C}
\end{array}$$

$h = \{\mathbf{ret} \ x \mapsto c, [\text{op}_j(k_i)_i \mapsto c_j]_j\}$

with h handle (ret v)

$$= c[v/x]$$

with h handle (op _{j} (c'_i)) _{i})

$$= c_j[\mathbf{fun} () \mapsto (\mathbf{with } h \text{ handle } c'_i)/k_i]_i$$

$h = \mathbf{handler} \{ \mathbf{ret } x \mapsto c, [\mathbf{op}_j(k_i)_i \mapsto c_j]_j \}$

ret 0

ret 0

= with goLeft handle (ret 0 \oplus ret 1)

ret 0

= with goLeft handle (ret 0 \oplus ret 1)

= with goLeft handle (ret 1 \oplus ret 0)

ret 0

= with goLeft handle (ret 0 \oplus ret 1)

= with goLeft handle (ret 1 \oplus ret 0)

= ret 1

ret 0

= with goLeft handle (ret 0 \oplus ret 1)

= with goLeft handle (ret 1 \oplus ret 0)

= ret 1



$$\left[T_1^h = T_2^h \right]_{(T_1 = T_2) \in \mathcal{E}_{\text{global}}}$$

h is correct

$$z_j^h = f_j : \text{unit} \rightarrow \underline{C}$$

$$\text{op}_j(T_i)_i^h = c_j[(\mathbf{fun} \ () \mapsto T_i^h)/k_i]_i$$

$$h = \{\mathbf{ret} \ x \mapsto c, [\text{op}_j(k_i)_i \mapsto c_j]_j\}$$

	<i>goLeft</i> <i>goRight</i>	<i>pickMax</i>	<i>sumAll</i>	<i>toList</i>	<i>toSet</i>
--	---------------------------------	----------------	---------------	---------------	--------------

$$z \oplus z = z$$



$$z_1 \oplus z_2 = z_2 \oplus z_1$$



$$(z_1 \oplus z_2) \oplus z_3 = z_1 \oplus (z_2 \oplus z_3)$$



$$\text{beep}(z_1) \oplus \text{beep}(z_2) = \text{beep}(z_1 \oplus z_2)$$



$$\text{fail}() \oplus \text{fail}() = \text{fail}()$$



$\oplus : 2$ $\text{beep} : 1$ $\text{fail} : 0$

$$z \oplus z = z$$

$$z_1 \oplus z_2 = z_2 \oplus z_1$$

$$(z_1 \oplus z_2) \oplus z_3 = z_1 \oplus (z_2 \oplus z_3)$$

$$\text{beep}(z_1) \oplus \text{beep}(z_2) = \text{beep}(z_1 \oplus z_2)$$

$$\text{fail}() \oplus \text{fail}() = \text{fail}()$$

$\oplus : 2$

beep : 1

fail : 0

for $\Sigma = \{\oplus : 2, \text{beep} : 1\}$, we have:

$$\begin{aligned} & \mathbf{do} \ x \leftarrow c \ \mathbf{in} \ (c_1 \oplus c_2) \\ & \quad = \\ & (\mathbf{do} \ x \leftarrow c \ \mathbf{in} \ c_1) \oplus (\mathbf{do} \ x \leftarrow c \ \mathbf{in} \ c_2) \end{aligned}$$

$$\begin{aligned} & \mathbf{do} \ x \leftarrow c_1 \ \mathbf{in} \ \text{beep}(c_2) \\ & \quad = \\ & \text{beep}(\mathbf{do} \ x \leftarrow c_1 \ \mathbf{in} \ c_2) \end{aligned}$$

$$\begin{aligned} & \mathbf{do} \ x_1 \leftarrow c_1 \ \mathbf{in} \ (\mathbf{do} \ x_2 \leftarrow c_2 \ \mathbf{in} \ c) \\ & \quad = \\ & \mathbf{do} \ x_2 \leftarrow c_2 \ \mathbf{in} \ (\mathbf{do} \ x_1 \leftarrow c_1 \ \mathbf{in} \ c) \end{aligned}$$

AN EFFECT SYSTEM FOR ALGEBRAIC EFFECTS AND HANDLERS

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ABSTRACT. We present an effect system for *core Eff*, a simplified variant of *Eff*, which is an ML-style programming language with first-class algebraic effects and handlers. We define an expressive effect system and prove safety of operational semantics with respect to it. Then we give a domain-theoretic denotational semantics of *core Eff*, using Pitts's theory of minimal invariant relations, and prove it adequate. We use this fact to develop tools for finding useful contextual equivalences, including an induction principle. To demonstrate their usefulness, we use these tools to derive the usual equations for mutable state, including a general commutativity law for computations using non-interfering references. We have formalized the effect system, the operational semantics, and the safety theorem in Twelf.

1. INTRODUCTION

An *effect system* supplements a traditional type system for a programming language with information about which computational effects may, will, or will not happen when a piece of code is executed. A well designed and solidly implemented effect system helps programmers understand source code, find mistakes, as well as safely rearrange, optimize, and parallelize code [11, 8]. As many before us [11, 24, 25, 7] we take on the task of striking just the right balance between simplicity and expressiveness by devising an effect system for *Eff* [2], an ML-style programming language with first-class algebraic effects [17, 15] and handlers [19].

Our effect system is *descriptive* in the sense that it provides information about possible computational effects but it does not prescribe them. In contrast, Haskell's monads *prescribe* the possible effects by wrapping types into computational monads. In the implementation we envision effect inference which never fails, although in some cases it may be uninformative. Of course, typing errors are still errors.

An important feature of our effect system is *non-monotonicity*: it detects the fact that a handler removes some effects. For instance, a piece of code which uses mutable state is determined to actually be pure when wrapped by a handler that handles away lookups and updates.

1998 ACM Subject Classification: D3.3, F3.2, F3.3.

Key words and phrases: algebraic effects, effect handlers, effect system.

A preliminary version of this work was presented at CALCO 2013, see [3].

AN EFFECT SYSTEM FOR ALGEBRAIC EFFECTS AND HANDLERS

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monads. However, when known handlers are used to handle operations, we may derive equivalences that describe the behavior of operations. The situation is opposite to that of [19], where we start with an equational theory for operations and require that the handlers respect it.

We demonstrate the technique for mutable state. Let $h = \text{state}_\iota$ and abbreviate

$\text{let } f = (\text{with } h \text{ handle } c) \text{ in } f e$

as $\mathcal{H}[c, e]$. Straightforward calculations give us the equivalences

$$\mathcal{H}[(\iota\#\text{lookup } ()) (y.c)), e] \equiv \mathcal{H}[c[e/y], e]$$

$$\mathcal{H}[(\iota\#\text{update } e' (-.c)), e] \equiv \mathcal{H}[c, e']$$

$$\mathcal{H}[\text{val } e' e] = \text{val } e'$$

by wrapping types into computational monads. In the implementation we envision effect inference which never fails, although in some cases it may be uninformative. Of course, typing errors are still errors.

An important feature of our effect system is *non-monotonicity*: it detects the fact that a handler removes some effects. For instance, a piece of code which uses mutable state is determined to actually be pure when wrapped by a handler that handles away lookups and updates.

1998 ACM Subject Classification: D3.3, F3.2, F3.3.

Key words and phrases: algebraic effects, effect handlers, effect system.

A preliminary version of this work was presented at CALCO 2013, see [3].

$$\mathcal{H}_{\max}[c] \stackrel{\text{def}}{=} \mathbf{with\ pickMax\ handle\ } c$$

$$\begin{aligned} & \mathcal{H}_{\max}[\mathbf{do\ } x \leftarrow c \mathbf{\ in\ } (c_1 \oplus c_2)] \\ & \quad = \\ & \mathcal{H}_{\max}[(\mathbf{do\ } x \leftarrow c \mathbf{\ in\ } c_1) \oplus (\mathbf{do\ } x \leftarrow c \mathbf{\ in\ } c_2)] \end{aligned}$$

$$\begin{aligned} & \mathcal{H}_{\max}[\mathbf{do\ } x_1 \leftarrow c_1 \mathbf{\ in\ } (\mathbf{do\ } x_2 \leftarrow c_2 \mathbf{\ in\ } c)] \\ & \quad = \\ & \mathcal{H}_{\max}[\mathbf{do\ } x_2 \leftarrow c_2 \mathbf{\ in\ } (\mathbf{do\ } x_1 \leftarrow c_1 \mathbf{\ in\ } c)] \end{aligned}$$

$$\mathcal{H}_{\text{sum}}[c] \stackrel{\text{def}}{=} \mathbf{with\ sumAll\ handle}\ c$$

$$\begin{aligned} & \mathcal{H}_{\text{sum}}[\mathbf{do}\ x \leftarrow c\ \mathbf{in}\ (c_1 \oplus c_2)] \\ & \qquad \qquad \qquad = \\ & \mathcal{H}_{\text{sum}}[(\mathbf{do}\ x \leftarrow c\ \mathbf{in}\ c_1) \oplus (\mathbf{do}\ x \leftarrow c\ \mathbf{in}\ c_2)] \end{aligned}$$

$$\begin{aligned} & \mathcal{H}_{\text{sum}}[\mathbf{do}\ x_1 \leftarrow c_1\ \mathbf{in}\ (\mathbf{do}\ x_2 \leftarrow c_2\ \mathbf{in}\ c)] \\ & \qquad \qquad \qquad = \\ & \mathcal{H}_{\text{sum}}[\mathbf{do}\ x_2 \leftarrow c_2\ \mathbf{in}\ (\mathbf{do}\ x_1 \leftarrow c_1\ \mathbf{in}\ c)] \end{aligned}$$

PART III

LOCAL

EFFECT **THEORIES**

values & computations

$v ::= \dots$

$c ::= \dots$

$h ::= \dots$

handlers

value types

$A ::= \dots$

$\underline{C} ::= A ! \Sigma / \mathcal{E}$

computation types

$$\left[T_1^h = T_2^h \right]_{(T_1 = T_2) \in \mathcal{E}_{\text{global}}}$$

h is correct

$$\frac{[T_1^h = T_2^h]_{(T_1 = T_2) \in \mathcal{E}}}{h \text{ respects } \mathcal{E}}$$

$$\Gamma \vdash h : A ! \Sigma \Rightarrow \underline{C}$$

h respects \mathcal{E}

$$\Gamma \vdash \mathbf{handler} \, h : A ! \Sigma / \mathcal{E} \Rightarrow \underline{C}$$

$$\frac{\Gamma \vdash v : A}{\Gamma \vdash \mathbf{ret} \, v : A ! \Sigma / \mathcal{E}}$$

$$\Gamma \vdash c_1 : A ! \Sigma / \mathcal{E} \quad \Gamma, x : A \vdash c_2 : B ! \Sigma / \mathcal{E}$$

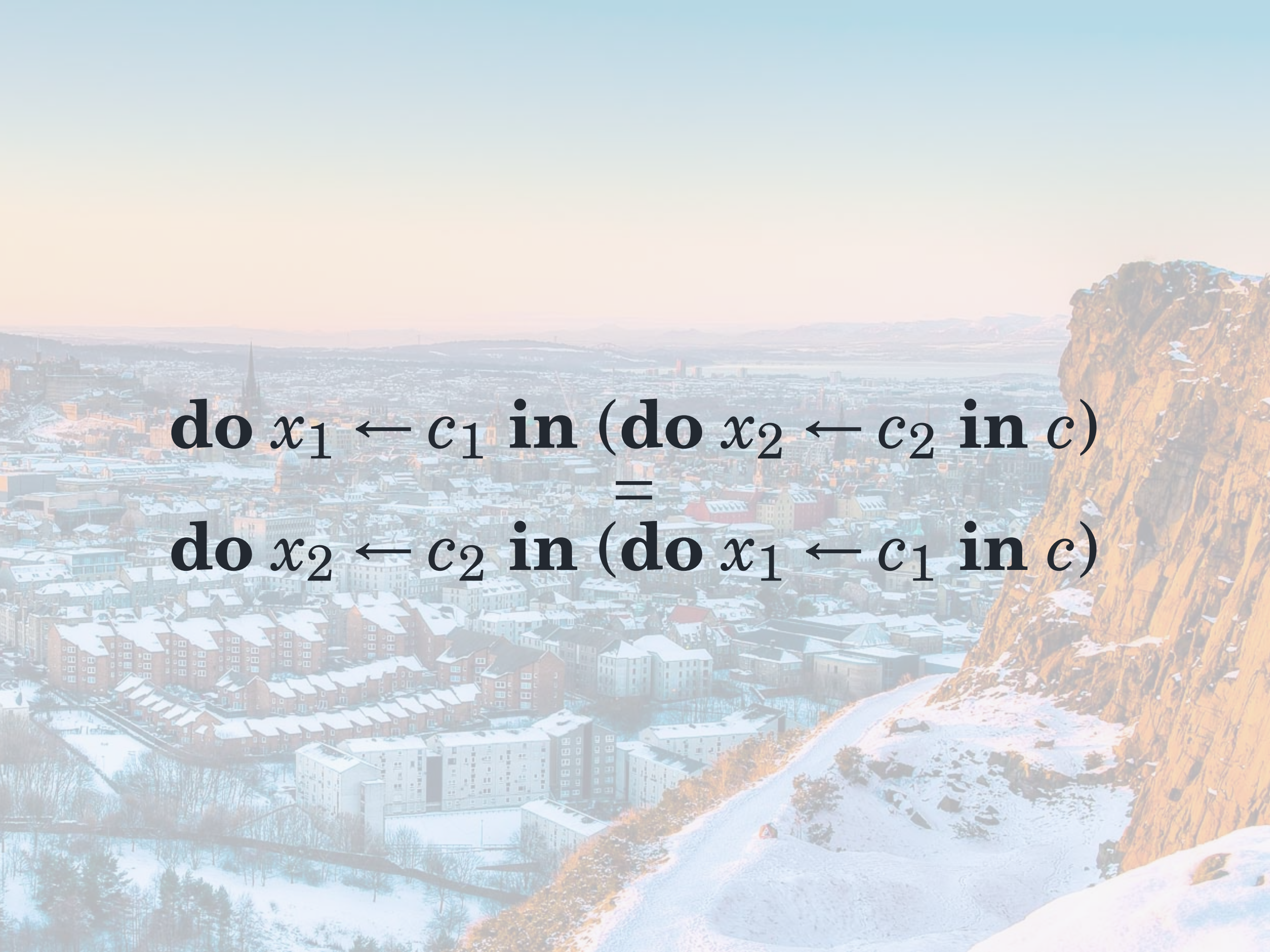
$$\Gamma \vdash \mathbf{do} \, x \leftarrow c_1 \, \mathbf{in} \, c_2 : B ! \Sigma / \mathcal{E}$$

$$\frac{[\Gamma \vdash c_i : A ! \Sigma]_i \quad \text{op}_i : k_i \in \Sigma}{\Gamma \vdash \text{op}(c_i)_i : A ! \Sigma / \mathcal{E}}$$

$$\frac{(T = T') \in \mathcal{E} \quad [c_i : A ! \Sigma / \mathcal{E}]_i}{T[c_i/z_i]_i =_{A! \Sigma / \mathcal{E}} T'[c_i/z_i]_i}$$

AN **EXAMPLE**





do $x_1 \leftarrow c_1$ in (do $x_2 \leftarrow c_2$ in c)
=
do $x_2 \leftarrow c_2$ in (do $x_1 \leftarrow c_1$ in c)

Algebraic Foundations for Effect-Dependent Optimisations

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Abstract

We present a general theory of Gifford-style type and effect annotations, where effect annotations are sets of effects. Generality is achieved by recourse to the theory of algebraic effects, a development of Moggi's monadic theory of computational effects that emphasises the operations causing the effects at hand and their equational theory. The key observation is that annotation effects can be identified with operation symbols.

We develop an annotated version of Levy's Call-by-Push-Value language with a kind of computations for every effect set; it can be thought of as a sequential, annotated intermediate language. We develop a range of validated optimisations (i.e., equivalences), generalising many existing ones and adding new ones. We classify these optimisations as structural, algebraic, or abstract: structural optimisations always hold; algebraic ones depend on the effect theory at hand; and abstract ones depend on the global nature of that theory (we give modularly-checkable sufficient conditions for their validity).

Categories and Subject Descriptors D.3.4 [Processors]: Compilers; Optimization; F.3.1 [Specifying and Verifying and Reasoning about Programs]: Logics of programs; F.3.2 [Semantics of Programming Languages]: Algebraic approaches to semantics; Denotational semantics; Program analysis; F.3.3 [Studies of Program Constructs]: Type structure

General Terms Languages, Theory.

Keywords Call-by-Push-Value, algebraic theory of effects, code transformations, compiler optimisations, computational effects, denotational semantics, domain theory, inequational logic, relevant and affine monads, sum and tensor, type and effect systems, universal algebra.

1. Introduction

In Gifford-style type and effect analysis [27], each term of a programming language is assigned a type and an effect set. The type describes the values the term may evaluate to; the effect set describes the effects the term may cause during its computation, such as memory assignment, exception raising, or I/O.

For example, consider the following term M :

if true then $x := 1$ else $x := \text{deref}(y)$

It has unit type 1 as its sole purpose is to cause side effects; it has effect set $\{\text{update}, \text{lookup}\}$, as it might cause memory updates or look-ups. Type and effect systems commonly convey this information via a type and effect judgement:

$x : \text{Loc}, y : \text{Loc} \vdash M : 1 ! \{\text{update}, \text{lookup}\}$

The information gathered by such effect analyses can be used to guarantee implementation correctness¹, to prove authenticity properties [15], to aid resource management [44], or to optimise code using transformations. We focus on the last of these. As an example, purely functional code can be executed out of order:

$x \leftarrow M_1; y \leftarrow M_2; N = y \leftarrow M_2; x \leftarrow M_1; N$

This reordering holds more generally, if the terms M_1 and M_2 have non-interfering effects. Such transformations are commonly used in optimising compilers. They are traditionally called *optimisations*, even if neither side is always the more optimal.

In a sequence of papers, Benton et al. [4–8] prove soundness of such optimisations for increasingly complex sets of effects. However, any change in the language requires a complete reformulation of its semantics and so of the soundness proofs, even though the essential reasons for the validity of the optimisations remain the same. Thus, this approach is not robust, as small language changes cause global theory changes.

A possible way to obtain robustness is to study effect systems in general. One would hope for a modular approach, seeking to isolate those parts of the theory that change under small language changes, and then recombining them with the unchanging parts. Such a theory may not only be important for compiler optimisations in big, stable languages. It can also be used for effect-dependent equational reasoning. This use may be especially helpful in the case of small, domain-specific languages, as optimising compilers are hardly ever designed for them and their diversity necessitates proceeding modularly.

The only available general work on effect systems seems to be that of Marino and Millstein [28]. They devise a methodology to derive type and effect frameworks which they apply to a call-by-value language with recursion and references; however, their methodology does not account for effect-dependent optimisations.

Fortunately, Wadler and Thiemann [46, 47] had previously made an important connection with the monadic approach to computational effects. They translated judgements of the form $\Gamma \vdash M : A ! \varepsilon$ in a region analysis calculus to judgements of the form $\Gamma' \vdash M' : T_\varepsilon A$ in a multi-monadic calculus. They gave the latter calculus an operational semantics, and conjectured the existence of a corresponding general monadic denotational semantics in which T_ε would denote a monad corresponding to the effects in ε , and in which the partial order of effect sets and inclusions would

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¹E. Cooper, S. Lindley, P. Wadler, and J. Yallop. Links 0.5, 2009. <http://groups.inf.ed.ac.uk/links>.



COMM

$$z_1 \oplus z_2 = z_2 \oplus z_1$$





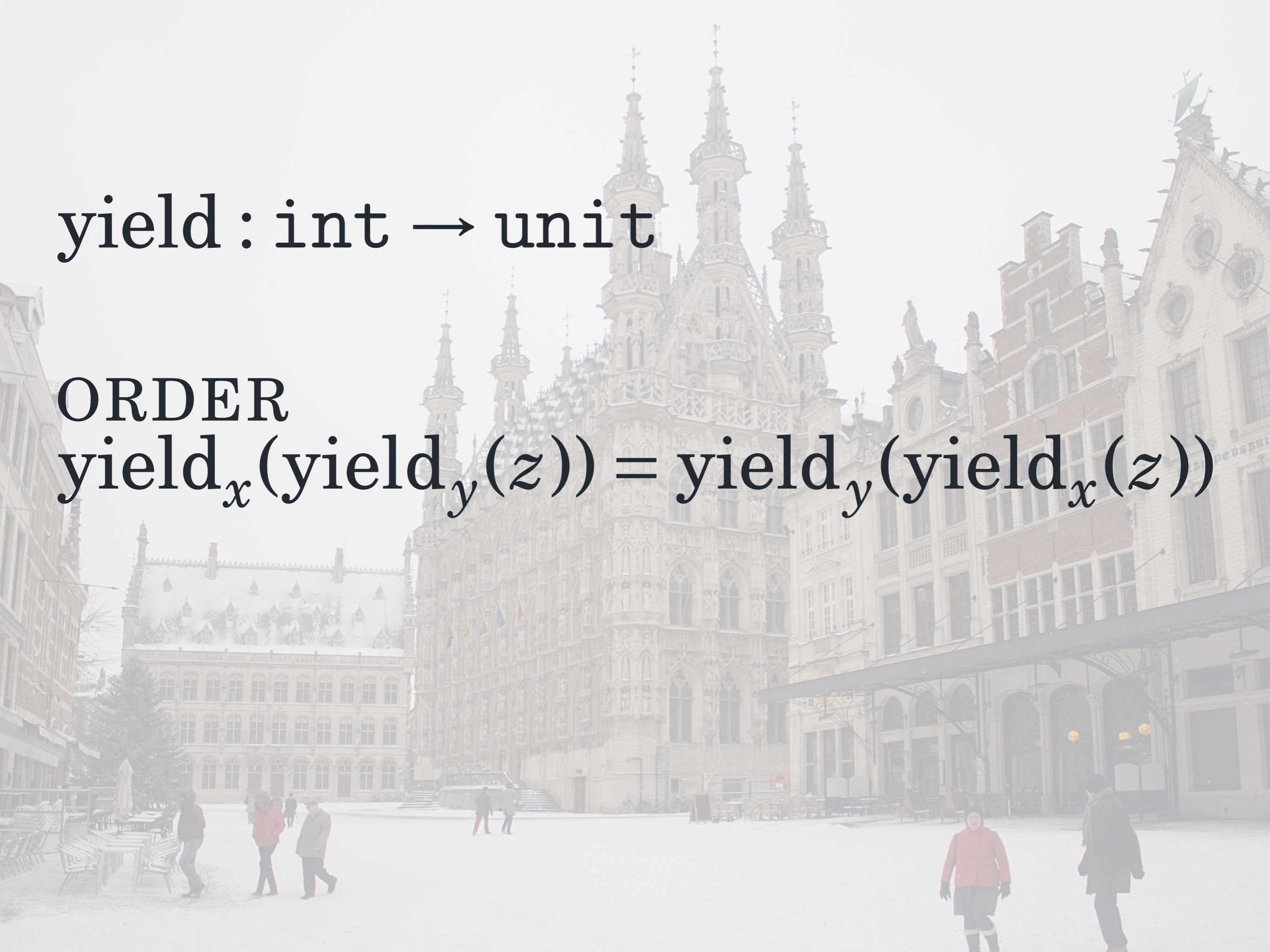
`yield : int → unit`



$\text{yield} : \text{int} \rightarrow \text{unit}$

ORDER

$\text{yield}_x(\text{yield}_y(z)) = \text{yield}_y(\text{yield}_x(z))$





$\text{yield} : \text{int} \rightarrow \text{unit}$

ORDER

$\text{yield}_x(\text{yield}_y(z)) = \text{yield}_y(\text{yield}_x(z))$

SUMYIELDED

$\text{unit}!\{\text{yield}\}/\{\text{ORDER}\} \Rightarrow \text{int}!\emptyset/\emptyset$





YIELDALL
handler {

$k_1 \oplus k_2 \mapsto k_1(); k_2()$

ret $x \mapsto \text{yield}_x()$

}

A background image of a snowy winter scene. A river flows through the center, with snow-covered banks and trees. A small bridge is visible in the distance. The overall tone is cold and serene.

YIELDALL
handler {

$k_1 \oplus k_2 \mapsto k_1(); k_2()$

$\text{ret } x \mapsto \text{yield}_x()$

}

$\text{int!}\{\oplus\}/\emptyset \Rightarrow \text{unit!}\{\text{yield}\}/\emptyset$

YIELDALL
handler {

$k_1 \oplus k_2 \mapsto k_1(); k_2()$

$\text{ret } x \mapsto \text{yield}_x()$

}

$\text{int}!\{\oplus\}/\emptyset \Rightarrow \text{unit}!\{\text{yield}\}/\emptyset$

$\text{int}!\{\oplus\}/\{\text{COMM}\} \Rightarrow \text{unit}!\{\text{yield}\}/\{\text{ORDER}\}$



$\text{int!}\{\oplus\}/\{\text{COMM}\}$



$\text{unit!}\{\text{yield}\}/\{\text{ORDER}\} \Rightarrow \text{int!}\emptyset/\emptyset$

$\text{int!}\{\oplus\}/\{\text{COMM}\} \Rightarrow \text{unit!}\{\text{yield}\}/\{\text{ORDER}\}$



in this talk

- put effect theories in types
- use equations to rewrite programs
- prove handlers respect equations

in the paper

- general operation signatures
- denotational semantics
- parameterized logic
- soundness theorems

in the next paper

- QuickCheck-like tool
- more examples
- compiler optimizations?